Short-range dependent processes subordinated to the Gaussian may not be strong mixing

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Abstract

There are all kinds of weak dependence. For example, strong mixing. Short-range dependence (SRD) is also a form of weak dependence. It occurs in the context of processes that are subordinated to the Gaussian. Is a SRD process strong mixing if the underlying Gaussian process is long-range dependent? We show that this is not necessarily the case.

Let $\{Z_i\}$ be a standardized Gaussian process with covariance function $\gamma(n) = n^{2H-2}L(n)$, where $1/2 < H < 1$ and $L(n)$ is slowly varying. We will consider instantaneous transformations $X_i = P(Z_i)$, where $\mathbb{E}P(Z_i)^2 < \infty$. The sequence $\{X_i\}$ is said to be LRD if the sum of its covariances diverges and SRD if the sum converges. Note that the sequence $\{Z_i\}$ is LRD because $\sum_{n=-\infty}^{+\infty} \gamma(n) = \infty$. The sequence $\{X_i\}$, however, may be LRD or SRD depending on $P(x)$.

Suppose now that $P(\cdot)$ is a finite-order polynomial. It can then be expressed as

$$P(x) = c_0 + \sum_{k=m}^{n} c_k H_k(x), \quad 1 \le m \le n,$$

with $c_m \neq 0$, where $H_k(x)$ is the k-th order Hermite polynomial. The bottom index m is called the *Hermite rank* of $P(x)$ and/or of the process $\{P(X_i)\}$.

It is known from Breuer and Major [1] that when

$$(2H-2)m + 1 < 0, \tag{1}$$

which can only happen when $m \ge 2$, then $\{X_i\}$ is SRD and as $N \to \infty$,

$$N^{-1/2} \sum_{i=1}^{[Nt]} [P(Z_i) - \mathbb{E}P(Z_i)] \xrightarrow{f.d.d.} \sigma B(t),$$

where $\sigma^2 = \sum_n \gamma(n)$, $B(t)$ is the standard Brownian motion and $\xrightarrow{f.d.d.}$ denotes convergence of finite-dimensional distributions. This seems to suggest that $\{P(Z_i)\}$ has weak dependence. It is natural to ask whether $\{P(Z_i)\}$ is strong mixing.[1] We will show that this may *not* be the case.

Key words long-range dependence, short-range dependence, Hermite rank, strong mixing.

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[1] A stationary process $\{X_i\}$ is said to be strong mixing if

$$\lim_{k\to\infty} \sup\left\{|P(A)P(B) - P(A\cap B)|,\ A \in \mathcal{F}_{-\infty}^{0}, B \in \mathcal{F}_{k}^{\infty}\right\} = 0,$$

where $\mathcal{F}_a^b$ is the σ-field generated by $X_a, \dots, X_b$.

Theorem 1. *Suppose that $\{Z_i\}$ is LRD with covariance $\gamma(n) = n^{2H-2}L(n)$, where H satisfies (1). The SRD process $\{X_i = P(Z_i)\}$ is not strong mixing if there exists a polynomial $Q(x)$ such that the Hermite rank m' of $Q(P(x))$ satisfies*

$$(2H-2)m' + 1 > 0. \tag{2}$$

Remark 2. The process $\{X_i = P(Z_i)\}$ in the theorem is SRD. The theorem states that this process is not strong mixing if there a polynomial $Q(x)$ such that the new process $\{Q(P(Z_i))\}$ is LRD. Note that (2) implies, in view of (1), that $m' < m$.

Proof. We argue by contradiction. Suppose that $\{X_i\}$ is strong mixing. Then by the definition of strong mixing, $\{Q(X_i)\}$ is also strong mixing. But (2) implies that (Taqqu [4])

$$s_N^2 := \mathrm{Var}\left[\sum_{i=1}^N Q(X_i)\right] \sim c_H L(N)^{m'} N^{(2H-2)m'+2}, \quad (2H-2)m'+2 > 1. \tag{3}$$

On the other hand, $S_N := \sum_{i=1}^N [Q(X_i) - \mathbb{E}Q(X_i)]$ is an element living on Wiener chaos of a finite order (see Janson [2], Chapter 2). By Janson [2], Theorem 5.10, for any $p > 2$, there exists a constant $c_p > 0$ depending only on p, such that

$$\mathbb{E}\left|s_N^{-1}S_N\right|^p \le c_p \left(\mathbb{E}\left|s_N^{-1}S_N\right|^2\right)^{p/2} = c_p.$$

Therefore $\{s_N^{-2}S_N^2, N \ge 2\}$ is uniformly integrable. By Theorem 1.3 of Peligrad [3], strong mixing and uniform integrability imply that

$$s_N^2 = l(N)N$$

for some slowly varying function $l(N)$. This contradicts (3). ☐

In some cases, no polynomial $Q(x)$ satisfies the requirement of Theorem 1. For example, when $P(x) = x^2$, then the Hermite rank $m = 2$, and one always has

$$\mathbb{E}Q(Z^2)H_1(Z) = \mathbb{E}Q(Z^2)Z = 0$$

for arbitrary polynomials $Q(x)$ (in fact for arbitrary $L^2(\Omega)$ functions). This is because $Q(Z^2)$ is an even function of Z. So the Hermite rank of $Q(P(x))$ is at least 2, and hence we don't have $m' < m$.

In the simple case where $P(x)$ is a Hermite polynomial, we have the following result:

Proposition 3. *Suppose $P(x) = H_m(x)$, $m \in \mathbb{Z}_+$. The polynomial $Q(x)$ required in Theorem 1 exists in either of the following cases:*

(a) $m \ge 4$ is even and $H > 3/4$.

(b) $m \ge 3$ is odd.

Proof. Using the product formula ((3.13) of Janson [2]) for Hermite polynomial, one has

$$H_m(x)^2 = \sum_{k=0}^m k!\binom{m}{k}^2 H_{2m-2k}(x), \tag{4}$$

$$H_m(x)^3 = \sum_{k_1=0}^m \sum_{k_2=0}^{(2m-2k_1)\wedge m} k_1!k_2!\binom{m}{k_1}^2\binom{2m-2k_1}{k_2}\binom{m}{k_2}H_{3m-2k_1-2k_2}(x). \tag{5}$$

For case (a), choose $3/4 < H < 1$, but not too big such that $\{P(X_i) = H_m(X_i)\}$ is SRD. This will happen by constraining H to satisfy (1). Now choose $Q(x) = x^2$. Then by (4),

$$Q(P(x)) = H_m(x)^2 = m! + (m-1)!m^2H_2(x) + \ldots,$$

so $\{Q(P(Z_i))\}$ has Hermite rank $m' = 2$, which is less than $m \geq 4$. Since $m' = 2$, and $H > 3/4$, we conclude that $\{Q(P(Z_i))\}$ is LRD and satisfies (2).

For case (b), choose $Q(x) = x^3$. Then

$$Q(P(x)) = H_m(x)^3 = a_1 H_1(x) + \dots$$

for some $a_1 > 0$. The term $H_1(x)$ appears when $3m - 2k_1 - 2k_2 = 1$, e.g., when $k_1 = (m-1)/2$, $k_2 = m$. The coefficient $a_1 > 0$ because all the coefficients before the Hermite polynomials in (5) are positive. It is then clear that the Hermite rank of $H_m(x)^3$ is $m' = 1$. Hence the polynomial $Q(x)$ satisfies (2). $\square$

Remark 4. In Proposition 3 case (b), we do not need a restriction on H. We require $m \geq 3$ since $m = 1$ is incompatible with (1).

Remark 5. What about the converse? Can a strong mixing process not be subordinated to a Gaussian LRD process? The answer is clearly "yes". Suppose for example $\{X_i\}$ i.i.d. Gaussian. Then there is no $\{X_i'\} \overset{f.d.d.}{=} \{X_i\}$ so that $X_i' = G(Z_i')$, where $\{Z_i'\}$ is LRD Gaussian, because the covariance $\mathrm{Cov}[X_i', X_0'] \neq 0$ for large i.

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